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Real Spinor Fields and the Electroweak Interaction

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The space of real spinor fields of a given mass $m > 0$ in Minkowski space is the direct sum of two irreducibly invariant subspaces under the connected Poincaré group **P**. These subspaces admit unique **P**-invariant positive-energy complex

Space reversal P and time reversal T act **P**-covariantly as antiunitary operators on the real fields. They also extend to the complex fields, in terms of which P exchanges the two subspaces, while T acts separately on each. When $m = 0$, the identical formalism produces a (rigorous) version of the conventional neutrino formalism.

The restriction to the real context of the conventional spinor-vector interaction is equivalent to the conventional $V - A$ interaction together with a computationally equivalent form of quantum electrodynamics in which the right electron and left electron are redundant.

1. INTRODUCTION

Particle theory has achieved a practical unification of quantum electrodynamics (QED) with weak interaction theory, via the "electroweak" theory of Glashow, Salam, and Weinberg developed in the 1960s. This theory incorporates, but does not explain, the experimental finding that parity is conserved in QED, but maximally broken in typical weak interactions. This incorporation involved cutting down the spinor-vector coupling to the "$V - A$" form found after many years of experimentation on the weak interactions, while leaving it intact in the case of QED. The present article shows that the $V - A$ form is a natural theoretical consequence of a simple unified formalism based on the use of real rather than complex spinor fields. Moreover, the same form may be applied to QED without any apparent experimental contraindication.

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The theory covers the interactions involving electron and the W-particle, and certain but not all interactions involving the Z-particle. While the "neutral current" interactions involving the Z originally appeared predominantly $V-A$, they now appear to involve additional particles, and will be treated here only peripherally. In terms of the more comprehensive particle theory indicated in [1], the present analysis treats the Minkowski space approximation (or deformation as the space curvature is allowed to tend to zero) to the interaction between the space of fermions of (conformal) weight 3/2 and the bosons of weight 1, in the universal cosmos (universal cover of the compactification of Minkowski space). Fermions of weight 5/2 as well as corresponding bosons may be involved in neutral current interactions, and are outside the scope of the present paper.

The apparent necessity of basing quantum mechanics on a complex (number) rather than real (number) formalism has intrigued physicists from the beginning of the subject. Stueckelberg, Feynman, and Schwinger are among many who have discussed the matter. The spinor formalism of Majorana [2] was an early concrete step in the real direction, but its use has been limited to neutrino and related studies.

Since direct physical measurements involve real rather than complex quantities, the role of complex numbers in quantum mechanics has appeared somewhat mysterious, while at the same time physically fundamental rather than merely a technical convenience. It is argued here that the basic problem with a purely real formalism is that it is inherently incapable of directly treating positivity of the energy.

Quite generally, real time evolution in a real vector space (e.g., as defined by an invariant wave equation) typically (and always, in the case of stable systems) involves complex conjugate eigenvalues for the infinitesimal generator of temporal evolution, or "pseudo-energy" $E\sharp$. For example, for an orthogonal one-parameter group in a real Hilbert space, the eigenvalues of $E\sharp$ are generically complex conjugate purely imaginary quantities. In order to have a parallel to stability for classical systems (where there is no ambiguity about the question of whether the hamiltonian is bounded below) it is essential to have (or to introduce) a notion of multiplication by i, so that the pseudo-energy $E\sharp$ (for which positivity is ambiguous) can be replaced by the presumptive observed energy E having real eigenvalues. E is defined as the corresponding operator iE in a complex space. This complex space is substantially of half the number of degrees of freedom, or dimension, of the real space. More precisely, if the $\{e_j\}$ form a basis in the complex space, then the $\{ie_j\}$ must be added to the $\{e_j\}$ to obtain the real basis.

As a simple example, consider the real Klein–Gordon equation $\Box\phi+m^2\phi=0$ in Minkowski space M, which may be considered to describe a neutral scalar field. The classical energy *functional* takes the form $\int[\frac{1}{2}(\text{grad}\,\phi)^2+\frac{1}{2}(\partial_o\,\phi)^2+m^2\phi^2]\,d_3x$, and is clearly positive and conserved. In order to obtain a quantum energy *operator*, one needs a notion of multiplication by i in the space $\mathbf{K}$ of all (normalizable, say) Klein–Gordon wave functions. An appropriate (relativistically invariant, positive-energy, etc.) notion is provided by the operation η: $\phi(k) \to i\theta(k)\,\phi(k)$, where $k = (k_o, k_1, k_2, k_3)$ denotes a point of momentum space, and $\theta(k) = \pm 1$ according as k_o is >0 or <0. To avoid undue circumlocution, the (technically elliptical) notation $\phi(k)$ denotes the value of the Fourier transform of ϕ at the point k. Note that in physical space M, η appears as a real but non-local operator, namely the Hilbert transform [3] with respect to time.

The relativistically invariant complex inner product between vectors in $\mathbf{K}$, involved in structuring $\mathbf{K}$ as a complex Hilbert space, has a simple local form in momentum space, but in physical space has nonlocal real part. Thus if $\langle \phi, \psi \rangle$ denotes the complex inner product of two real wave functions ϕ and ψ, then $Re\{\langle \phi, \psi \rangle\} = \langle D\phi, D\psi \rangle_2 + \langle D^{-1}\partial_o \phi, D^{-1}\partial_o \psi \rangle_2$ where D denotes the operator $(m^2 + \Delta)^{1/4}$ and $\langle f, g \rangle_2$ denotes the L_2 inner product $\int f(x)\, g(x)^*\, d_3 x$.

Similarly, the action of the complex unit i in physical space is nonlocal: it carries the wave function having the initial data $\phi(0, \mathbf{x}) = f(\mathbf{x})$, $\partial_o \phi(0, \mathbf{x}) = g(\mathbf{x})$, at time 0 into the wave function having the corresponding data $-D^{-1}g$ and Df. It thus is not at all multiplication by the "number" $(-1)^{1/2}$. On the other hand, the imaginary part of the inner product, which essentially defines the kernel of the commutator function for the quantized field, is entirely local in M as well as in momentum space: $Im\{\langle \phi, \psi \rangle\} = \langle \phi, \partial_o \psi \rangle_2 - \langle \partial_o \phi, \psi \rangle_2$.

η evidently has the essential properties that $\eta^2 = -1$, and that $\eta U(g) = U(g)\eta$ if g is any transformation of the connected Poincaré group $\mathbf{P}$, and $U(g)$ denotes its action in $\mathbf{K}$; and that $U(g)$ is unitary in $\mathbf{K}$ as a complex Hilbert space with complex structure η. The energy operator, which acts simply as multiplication by $k_o\theta(k)$ in momentum space is evidently positive. It can be shown that η is the only such operator. The example just given is adapted from [5].

Thus even in this simple case, the elementary, local complex structure consisting of multiplication by i on wave functions in configuration space, is not at all the complex structure appropriate to the quantized field, which is rather η. Superficially it might appear such since in momentum space, the complex structure is multiplication by $i\theta(k) = \pm i$, but this is misleading as shown by consideration of the more general case of the real Klein–Gordon equation with an external time independent non-negative potential $V(x)$. In this case the complex structure appropriate to the quantized field is given in configuration space by the same operator as above with D replaced by $[(m^2 - \Delta) + V]^{1/4}$, which is nonlocal in momentum as well as physical space. In both cases D is in fact antilocal, in the sense that it carries functions that vanish on a given set into functions that are nonzero almost everywhere on the set (see, e.g., [6] for the case $V=0$). Relatedly, in the case of the Klein–Gordon equation (with or without an external potential), the commutator in the corresponding quantized field is entirely local in the test functions for the field operators, while the 2-point function is antilocal, as in the heuristic considerations of Reeh and Schlieder [7].

The foregoing construction of an appropriate Hilbert space of complex fields starting from a given space of real fields—specifically, the single-particle space of the corresponding quantized field—will be adapted below to the case of spinor fields. Traditionally, the given local complex structure, of multiplication by i on wave functions in physical space, has been used in connection with complex spinor fields and the Dirac equation. This is convenient algebraically, but leads to difficulties in the treatment of negative frequency components. These difficulties led to the hole theory of Dirac, and then later, in a form generally regarded as more satisfactory, to the interchange "by hand" of creation and annihilation operators in the negative frequency sector of the associated quantized field, in order to obtain the necessary positive energy.

The switch between creation and annihilation operators is, however, equivalent to a change of complex structure. The latter consists, more specifically, of the replacement of simple multiplication by i on configuration space wave functions, by the operation of multiplication by i on positive-frequency wave functions and multiplication by $-i$ on negative frequency wave functions. This is the same nonlocal (in configuration space) complex structure that is involved in the case of the real Klein–Gordon equation treated above. Quantization in the presence of a given external potential requires a more complicated complex structure, if energy-positivity and a nontrivial vacuum for the corresponding quantized field are required, as in the case of the Klein–Gordon equation. A theorem due originally to M. Weinless (e.g., [4]) shows that the requisite complex structure is in fact unique.

In the case of real spinor fields, two simple invariant complex structures will be used in this space of fields to pick out two irreducibly **P**-invariant subspaces. These subspaces will then be endowed with positive-energy complex Hilbert space structures. In the massive case, these Hilbert spaces are unitarily and covariantly equivalent to the usual "left" electron and "right" positron subspaces. In the massless case, they are similarly related to the usual neutrino and antineutrino Hilbert spaces.

The adaptation of the usual trilinear fermion–antifermion–vector interaction to the real spinor formalism provides a theory whose empirical implications appear identical (modulo experiments to date) to those of QED and of electroweak charged current theory. The basis for the $V-A$ interaction proposed early on by Feynman & Gell-Mann and independently by Marshak & Sudarshan in 1958, to account for the parity nonconservation in the weak interactions established by Wu *et al.* in 1957, following the watershed work of Lee and Yang in 1956 questioning this conservation, is thereby clarified. In the real formalism in fact, the $V-A$ interaction appears as invariant under space reversal P, and the electron, like the neutrino, is carried into its antiparticle by P. No invariant operator in local spin spaces such as charge conjugation C exists in the real formalism, providing a substantial analytic simplification. The absence of C in the real formalism marks the fundamental difference of the real formalism from the CP proposal of Landau in 1957, which is conventional in having twice as many single-particle modes as the present formalism.

More specifically, in the general complex spinor field, there are four distinct "species" of a given mass $m \geqslant 0$, where mass is defined as the minimal (positive) energy in all Lorentz frames. Here "species" means an irreducibly invariant subspace of fields under the connected Poincaré group $\mathbf{P}$, with a one-sided frequency spectrum. These consist of the ranges of the projections $F^{\pm}P^{\pm}$, where $F^{\pm}$ are the projections onto the positive/negative frequency subspaces and $P^{\pm}$ are the projections $(1 \pm \gamma_5)/2$. The real spinor fields however provide only two such species, and these two species suffice to describe the observed free electron or neutrino states, as labeled by the usual quantum numbers of energy-momentum and helicity.

The phenomenon of parity nonconservation has led via the $V-A$ interaction to the elimination of two of the four complex spinor species in connection with charged current weak interactions, and to their partial elimination in the case of neutral current interactions. This is naturally suggestive of the conceivable redundancy of half of the complex spinor field modes. However, in the Maxwell–Dirac equations, the right and left electrons play symmetric roles. This situation naturally presents the question of why QED is parity-invariant while the weak interactions are not.

Although the present real formalism involves only two species, it is covariant under the full Poincaré group, including discrete symmetries, and both species are of positive energy. The modified QED and charged current weak interactions are essentially identical in form. Notwithstanding the great mathematical importance of the Dirac equation and operator, they lack direct physical interpretations; rather, the equation was originally and is still argued for on formal grounds. The present massive fields satisfy instead a second-order equation expressing the directly physical fact that the electron and positron are mass eigenstates. This together with the Poincaré covariance of the fields is quite sufficient for physical purposes. In the real formalism, the Dirac operator may be characterized as the essentially unique covariant operator that interchanges particle and antiparticle. Unlike C, however, it does not operate only on the spin spaces at individual points of space-time, but involves also a differential operator.

2. COMPLEX STRUCTURES IN REAL SPINOR FIELDS

Our starting point is not a *given wave equation*, such as that proposed by Dirac in 1928, from which invariance features are derived, but rather a *given relativistically invariant space of fields in Minkowski space*. The transformation properties of these fields under temporal evolution are included in a global form among the transformation properties under the Poincaré group. By differentiation a corresponding differential equation results, and need not be postulated a priori.

Let **S** denote the space of all *real* spinor fields on Minkowski space M whose Fourier transforms are supported on the hyperboloid $k^2 = m^2$, where m is a given non-negative number. These are four-component real fields that transform according to the *real* form of the spin representation of the Lorentz group. Specifically, this representation carries the generator $L_{\mu\nu}(\mu, \nu = 0, 1, 2, 3)$ of the group into the real matrix $\gamma_\mu \gamma_\nu$, where the gammas are taken in the Majorana representation:

$$\gamma_o = \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix},$$

$$\gamma_2 = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \qquad \gamma_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}.$$

The following are **P**-invariant complex structures in **S**.

1. Let ω denote $\gamma_0 \gamma_1 \gamma_2 \gamma_3$. Then ω is a **P**-invariant complex structure in S.

Since $\omega = -i\gamma_5$, this is evident. Of course, ω also defines an invariant complex structure in spin space by itself, since it has no nontrivial action on the underlying space-time. A related invariant local complex structure, in a space of which the present one is a quotient, was given in [1]; the specific form given here was suggested by I. T. Todorov.

2. Let η denote the Hilbert transform with respect to time. Specifically, η carries an arbitrary wave function $\psi(k)$ in momentum space $M^\wedge$ into $i\theta(k)\,\psi(k)$. Then η is also a **P**-invariant complex structure in S.

This is a well-known property of the Hilbert transform, and of its action on momentum-space wave functions. As noted above, η also defines a complex structure in real scalar fields; it has no nontrivial action in spin space.

It will be shown that **S** is covariantly and unitarily equivalent to the direct sum of the usual left-electron and right-positron complex Hilbert spaces, in the massive case, and has an analogous decomposition in the massless case. To this end, note first that there is defined on the space C^4 of complex 4-tuples a canonical nontrivial invariant antihermitian form under the *real* spin representation of the connected Lorentz group **L**. For arbitrary vectors u and v in C^4 let the positive definite inner product $\sum_\mu u_\mu v_\mu^*$ (here and henceforth μ and ν range over the values 0, 1, 2, 3, and * denotes complex conjugation) be denoted as $\langle u, v \rangle$. For arbitrary wave functions ψ, $\psi(x)$ will denote the value at the point x of configuration space M, and $\psi(k)$ at the point k of momentum space.

(i) The sesquilinear form on C^4: $\langle\langle u, v \rangle\rangle = \langle \gamma_0 u, v \rangle$, is invariant under the real spin representation of **L**, antihermitian, and nondegenerate.

Specifically, $\langle \gamma_0 u, v \rangle = u_0 v_3^* - u_3 v_0^* + u_2 v_1^* - u_1 v_2^*$. To show the **L**-invariance of the form $\langle\langle u, v \rangle\rangle$, note that this means that $\langle\langle Xu, v \rangle\rangle + \langle\langle u, Xv \rangle\rangle = 0$ if X represents the action of an arbitrary generator of **L** in the real spin representation. A basis for the generators (see below) acts by the matrices $\gamma_\mu \gamma_\nu (\mu \neq \nu)$. If $\mu = 0$ and $\nu = j$ (where here and henceforth j or k range over the values 1, 2, 3), then noting that the γ_j are hermitian while γ_0 is antihermitian, it follows that

$$\begin{aligned} \langle\langle \gamma_0 \gamma_j u, v \rangle\rangle + \langle\langle u, \gamma_0 \gamma_j \rangle\rangle &= \langle \gamma_0 \gamma_0 \gamma_j u, v \rangle + \langle \gamma_0 u, \gamma_0 \gamma_j v \rangle \\ &= -\langle \gamma_j u, v \rangle + \langle -\gamma_0^2 u, \gamma_j v \rangle \\ &= -\langle \gamma_j u, v \rangle + \langle u, \gamma_j v \rangle = 0. \end{aligned}$$

If neither of μ and ν is 0, then similarly

$$\begin{aligned} \langle\langle \gamma_j \gamma_k u, v \rangle\rangle + \langle\langle u, \gamma_j \gamma_k v \rangle\rangle &= \langle \gamma_0 \gamma_j \gamma_k u, uv \rangle + \langle \gamma_0 u, \gamma_j \gamma_k v \rangle \\ &= \langle \gamma_0 \gamma_j \gamma_k u, v \rangle - \langle \gamma_j \gamma_k \gamma_0 u, v \rangle \\ &= \langle \gamma_0 \gamma_j \gamma_k u, v \rangle - \langle \gamma_0 \gamma_j \gamma_k u, v \rangle = 0. \end{aligned}$$

It is evident that the form $\langle\langle u, v \rangle\rangle$ is antihermitian (i.e., $\langle\langle y, x \rangle\rangle = -\langle\langle x, y \rangle\rangle^*$) and nondegenerate.

(ii) Let χ denote the operator $\psi(k) \to -(\sum \gamma_\mu k_\mu)\, \theta(k)\, \psi(k)$. Then: (a) χ leaves the space S invariant; (b) the form $\langle\langle \chi u, v \rangle\rangle$ is hermitian: $\langle\langle \chi u, v \rangle\rangle = \langle\langle u, \chi v \rangle\rangle$; (c) $\langle\langle \chi u, u \rangle\rangle \geqslant 0$ for arbitrary u.

Ad(ii-a), note that a wave function ψ has real values on M if and only if it has the property $\psi(-k) = \psi(k)^*$ in momentum space. The application of χ preserves this property since $(\sum_\mu \gamma_\mu k_\mu)\, \theta(k)$ is real and invariant under the map $k \to -k$.

Ad(ii-b), note that

$$\begin{aligned}
\langle\!\langle \chi u, v\rangle\!\rangle &= \left\langle \gamma_0 \left(\sum \gamma_\mu k_\mu\right) \theta(k) u, v \right\rangle \\
&= -\left\langle \left(\sum \gamma_\mu k_\mu\right) \theta(k) u, \gamma_0 v \right\rangle \\
&= -\left\langle \gamma_0 \theta(k) v, \left(\sum \gamma_\mu k_\mu\right) u \right\rangle^* \\
&= -\left\langle \left(-\gamma_0 k_0 + \sum \gamma_j k_j\right) \gamma_0 \theta(k)\, v, u \right\rangle^* \\
&= \left\langle \gamma_0 \left(\sum \gamma_\mu k_\mu\right) \theta(k)\, v, u \right\rangle^* = \langle\!\langle \chi v, u\rangle\!\rangle^*.
\end{aligned}$$

Ad(ii-c), note that when $m > 0$ it suffices by Lorentz invariance to consider the case in which all $k_j = 0$. In this case $\langle\!\langle \chi u, v\rangle\!\rangle = -\langle \gamma_0 \theta(k)\, \gamma_0 k_0 u, u\rangle = \langle |k_0|\, u, u\rangle \geqslant 0$. When $m = 0$, it suffices for the same reason to consider the case in which $k_0 = k_1$, $k_2 = k_3 = 0$. Then $\langle\!\langle \chi u, u\rangle\!\rangle = -\langle\!\langle \gamma_0(\gamma_0 k_0 + \gamma_1 k_0)\, \theta(k)\, u, u\rangle\!\rangle = -|k_0|\, \langle \gamma_0(\gamma_0 + \gamma_1)\, u, u\rangle = |k_0|\, \langle (1 - .\gamma_0\gamma_1)\, u, u\rangle = |k_0|\, [(u_0 - u_2)(u_0 - u_2)^* + (u_1 + u_3)(u_1 + u_3)^*] \geqslant 0$.

(iii) For arbitrary real spinor wave functions ϕ and ψ, define the inner product

$$\langle \phi, \psi\rangle = \int \langle\!\langle \chi\phi(k), \psi(k)\rangle\!\rangle \, d\Lambda(k),$$

where the integration is over the mass hyperboloid $k^2 = m^2$, $d\Lambda(k)$ denotes the element of Lorentz-invariant measure $d_3 k/|k_0|$, and $\langle \phi, \psi\rangle$ is undefined unless the integral is convergent. A vector ϕ such that $\langle \phi, \phi\rangle < \infty$ is called normalizable. Let $\mathbf{S}_\pm$ denote the spaces of all normalizable real spinor fields ψ with the property that $\eta\omega\psi = \pm\psi$. Let $\mathbf{H}^\pm$ denote the respective spaces of wave functions in $\mathbf{S}_\pm$, where the inner product is as given, and the complex structure giving the action of i in $\mathbf{H}^\pm$, is defined as $\pm\omega$ in the momentum space realization. Then $\mathbf{H}^\pm$ is a complex Hilbert space.

In fact, denoting the integrand in the above expression as $\langle \psi, \psi'\rangle$, it is easily checked that this is complex-linear in ψ, complex-antilinear in ψ', and non-negative: $\langle \psi, \psi\rangle \geqslant 0$. In the case $m > 0$, $\langle\!\langle \chi u, u\rangle\!\rangle = 0$ if and only if $u = 0$, implying that $\langle \psi, \psi\rangle = 0$ only if $\psi = 0$. When $m = 0$, $\langle\!\langle \chi u, u\rangle\!\rangle$ may vanish for nonvanishing u, but if ψ is a nonzero vector in $\mathbf{S}_\pm$, $\langle \psi, \psi\rangle$ must be strictly positive. To show this, consider the case of $\mathbf{S}_+$ (that of $\mathbf{S}_-$ is similar). The momentum space form of ψ then has the property that

$\omega\psi(k) = i\psi(k)$ if $k_0 > 0$. The computation above under (ii) shows that for $\langle\langle \omega u, u \rangle\rangle$ to vanish, it is necessary that $u_0 = u_2$ and $u_1 = -u_3$. If $u = \psi(k)$, this property is incompatible with the property that $\omega u = iu$ unless $u = 0$. The inner product $\langle \psi, \psi' \rangle$ together with the given complex structures thus determine fully equipped complex Hilbert spaces.

(iv)(a) The $\mathbf{S}_{\pm}$ are $\mathbf{P}$-invariant, and invariant under χ;

(b) $\mathbf{S}$ is the direct sum of $\mathbf{S}_-$ and $\mathbf{S}_+$;

(c) The action $U_{\pm}$ of $\mathbf{P}$ on $\mathbf{S}_{\pm}$ is an irreducible unitary

representation of $\mathbf{P}$, of positive energy. U_+ is unitarily equivalent to the action of $\mathbf{P}$ on positive energy, positive frequency complex spinors fields ψ of the same mass such that $(1 + \gamma_5)\,\psi = 0$, via the mapping from such complex fields to the real fields that coincide in momentum space at positive frequencies. U_- is unitarily equivalent to the action of $\mathbf{P}$ on positive energy (negative frequency) complex spinor fields such that $(1 - \gamma_5)\,\psi = 0$, having the same mass. When $m > 0$, $\mathbf{S}_+$ and $\mathbf{S}_-$ are respectively $\mathbf{P}$-covariantly unitarily equivalent to the (positive-energy) left-electron and right-positron Hilbert spaces. When $m = 0$, they are similarly equivalent to the neutrino and antineutrino Hilbert spaces.

Ad(a), note that since ω and η commute, $(\omega\eta)^2 = \omega^2\eta^2 = (-1)(-1) = 1$. Moreover, in the momentum space representation, each of ω and η is $\mathbf{P}$-invariant. It follows that their product is $\mathbf{P}$-invariant, and that the eigenspaces $S_{\pm}$ are correspondingly $\mathbf{P}$-invariant.

Ad(b), this means that every vector in $\mathbf{S}$ is the sum of unique vectors in $\mathbf{S}_+$ and in $\mathbf{S}_-$. The projections $P_{\pm} \equiv (1 \pm \eta\omega)/2$ evidently carry $\mathbf{S}$ into $\mathbf{S}_{\pm}$, and have product 0. If x is an arbitrary vector in $\mathbf{S}$, it follows that $x = P_+ x + P_- x$, and that this is the unique decomposition of x as the sum of vectors in $\mathbf{S}_{\pm}$.

Ad(c), let ϕ be arbitrary in $\mathbf{S}_-$, and let ψ denote the complex spinor field of the same mass whose positive-frequency component is identical to ϕ, and whose negative-frequency component vanishes. The mapping $R\colon \phi \to \psi$ is then relativistically covariant: $V(g)\, R\phi = RU(g)\phi$, where U and V denote the actions of the arbitrary transformation g in P on real and complex spinor fields (resp.). The claim is that $R(\mathbf{S}_-)$ consists precisely of the "left" complex spinors of positive energy, i.e. the complex positive-frequency spinors ψ such that $(1 + \gamma_5)\,\psi = 0$.

To see this, note that vectors ϕ in $\mathbf{S}_-$ are characterized by the property that $\eta\omega\phi = -\phi$, i.e. $(1 + \eta\omega)\phi = 0$. It follows that in momentum space, in the region where $k_0 > 0$, $(1 + i\omega)\,\phi(k) = 0$. Since $i\omega = \gamma_5$, this means that $(1 + \gamma_5)\,\phi = 0$, i.e. ϕ is a "left" particle. Thus $\mathbf{S}_-$ is covariantly equivalent to the positive-energy left complex spinor fields of the same mass. The latter fields form an irreducible unitary positive-energy representation of $\mathbf{P}$, implying that the same is true of $\mathbf{S}_-$. In the case of $\mathbf{S}_+$, the argument is the same except that the negative rather than positive frequencies are used.

To look at this in a slightly different way, note that the subspace of complex spinor fields ϕ defined by the equation $(1+\gamma_5)\,\phi=0$ corresponds in the Majorana momentum space representation to the wave functions ψ for which $(1+i\omega)\,\psi(k)=0$, or $\omega\psi(k)=i\psi(k)$, when $k_0>0$. Real spinor fields ψ in M are uniquely determined by their positive-frequency components in momentum space, by virtue of the property that $\psi(-k)=\psi(k)^*$. Now multiplication by i on the positive frequency subspace requires multiplication by $-i$ on the negative frequency subspace, in order to leave invariant the space of real spinors. This operator is the Hilbert transform η with respect to time. Thus the positive energy "left" electron space corresponds to $\mathbf{S}_-$. Similarly the corresponding space of negative frequency (positron) spinors is antiunitarily equivalent to the subspace $\mathbf{S}_+$ (i.e., unitarily equivalent after changing the sign of the frequency, or equivalently of the complex structure in the applicable space).

3. DISCRETE SYMMETRIES

We next consider the action of discrete symmetries on $\mathbf{S}$, from the standpoint of general group theory. Any discrete symmetry d such as space or time inversion defines a transformation on $\mathbf{P}$: $g \to d^{-1}\,gd$. There is then no element h of $\mathbf{P}$ such that $d^{-1}\,gd=h^{-1}\,gh$. For any given representation R of $\mathbf{P}$ on a linear vector space L, there may or may not exist a transformation D on L such that $R(d^{-1}\,gd)=D^{-1}R(g)\,D$. When D exists, the discrete symmetry is *implementable* in the representation V.

Space and time inversion are in fact implementable on the space $\mathbf{S}$. To show this, note that both space and time inversion have the following action on the Lorentz group generators $L_{\mu\nu}$:

$$L_{0\mu} \to -L_{0\mu}; \qquad L_{jk} \to L_{jk}.$$

As earlier noted, $L_{\mu\nu}$ acts as $\gamma_\mu\gamma_\nu$ on real spin space. It follows that the action on spin space of space inversion may be represented by $\gamma_0\omega$. The factor ω is needed here to insure that the action of space inversion leaves invariant the decomposition of $\mathbf{S}$ as $\mathbf{S}_+ \oplus \mathbf{S}_-$, while interchanging $\mathbf{S}_+$ and $\mathbf{S}_-$. More specifically, if P denotes the geometrical action of space inversion in M, i.e. P: $x_0 \to x_0$, $x_j \to -x_j$, then the action of P is $U(P)$:

$$\psi(x) \to \gamma_0\omega\psi(Px).$$

In momentum space, the corresponding action is

$$\psi(k) \to \gamma_0 \omega \psi(Pk),$$

where P denotes also the contragredient action $k_0 \to k_0, k_j \to -k_j$. The action of $U(P)$ on $\psi(k)$ is to carry it into $\gamma_0 \omega \psi(Pk)$. Note that $\omega\eta$ anticommutes with $U(P)$, since

$$\omega\eta U(P)\colon \psi(k) \to \omega\eta\gamma_0\omega\psi(Pk) = -\eta\gamma_0\omega^2\psi(Pk) = \eta\gamma_0\psi(Pk)$$

while

$$U(P)\, \omega\eta\colon \psi(k) \to \gamma_0\omega\omega\eta\psi(Pk) = -\gamma_0\eta\psi(Pk) = -\eta\gamma_0\psi(Pk),$$

where the invariance of $\theta(k)$ under P is used.

Hence if ψ is in $\mathbf{S}_+$, i.e. $\omega\eta\psi = \psi$, then $\omega\eta U(P)\, \psi = -U(P)\, \omega\eta\psi = -U(P)\, \psi$, showing that $U(P)\, \psi$ is in $\mathbf{S}_-$. Similarly, if ψ is in $\mathbf{S}_-$, then $U(P)\, \psi$ is in $\mathbf{S}_+$.

In the case of time reversal T, $U(T)$ acts in M as $\psi(x) \to \gamma_0\omega\psi(Tx)$, and in momentum space as $\psi(k) \to \gamma_0\omega\psi(Tk)$. In contrast to the case of space reversal, $U(T)$ commutes with $\omega\eta$. Thus

$$\omega\eta U(T)\colon \psi(k) \to \omega\eta\gamma_0\omega\psi(Tk) = -\omega^2\eta\gamma_0\psi(Tk) = \eta\gamma_0\psi(Tk)$$

while

$$U(T)\, \omega\eta\colon \psi(k) \to \gamma_0\omega\omega(-\eta)\, \psi(Tk) = \gamma_0\eta\psi(Tk)$$

since $\theta(Tk) = -\theta(k)$.

It follows that $U(T)$ leaves $\mathbf{S}_+$ and $\mathbf{S}_-$ separately invariant.

When $m > 0$, this contrasts with the conventional actions of P and T, which respectively map particles into particles and antiparticles, whereas here they do the opposite. This difference is however basically semantic and devoid of physically measurable implications. However, it shows that "parity" in the sense of the action of space inversion on the underlying fields is different in the present real formalism from the conventional complex formalism.

Remark. This formalism produces directly the positive energy representations (in $\mathbf{S}_+$ and $\mathbf{S}_-$), corresponding to the single particle subspace of the positive-energy quantized field. The conventional formalism based on the Dirac equation results in both positive- and negative- energy representations in the single-particle space. This necessitates ad hoc conventions, such as hole theory, or the interchange of creation and annihilation operators (which is equivalent to changing the complex structure by multiplication by $\theta(k)$ in momentum space), in order to attain the physically essential positive energy states. The present setup produces the same (unitarily equivalent) quantized field, without the use of any such devices.

A further contrast between the real and the conventional complex formalisms is that P, as well as T, acts in an anti-unitary fashion in $\mathbf{H}$. To show the anti-unitarity of $U(P)$ and $U(T)$, observe to begin with that $U(P)$ and $U(T)$ anticommute with χ. In the case of P, to compute $U(P)\,\chi\psi$, recall that $\chi\psi = -(\sum_\mu \gamma_\mu k_\mu)\,\theta(k)\,\psi(k)$, whence

$$U(P)\,\chi\psi\colon \psi \to -\gamma_0\omega(\gamma_0 k_0 - \gamma_1 k_1 - \gamma_2 k_2 - \gamma_3 k_3)\,\theta(k)\,\psi(Pk)$$

since $\theta(Pk) = \theta(k)$. In the opposite order,

$$\begin{aligned}\chi U(P)\colon \psi(k) &\to -\left(\sum_\mu \gamma_\mu k_\mu\right)\theta(k)\,\gamma_0\omega\psi(Pk)\\ &= \gamma_0(\gamma_0 k_0 - \gamma_1 k_1 - \gamma_2 k_2 - \gamma_3 k_3)\,\omega\theta(k)\,\psi(Pk)\\ &= \gamma_0\omega(\gamma_0 k_0 - \gamma_1 k_1 - \gamma_2 k_2 - \gamma_3 k_3)\,\theta(k)\,\psi(Pk),\end{aligned}$$

showing that $U(P)\,\chi = -\chi U(P)$.

A complex-antilinear transformation in Hilbert space that leaves invariant the real part of the inner product is automatically antiunitary. To show that $U(P)$ is antiunitary it therefore suffices to show that it leaves invariant $\int \langle\!\langle \chi\psi(k), \psi'(k)\rangle\!\rangle\, d\Lambda(k)$. Under $U(P)$ this form becomes

$$\begin{aligned}&\int \langle\!\langle \chi U(P)\,\psi(k), \psi'(k)\rangle\!\rangle\, d\Lambda(k)\\ &\quad = \int \langle \gamma_0\chi\gamma_0\omega\psi(Pk), \gamma_0\omega\psi'(Pk)\rangle\, d\Lambda(k)\\ &\quad = \int \left\langle \gamma_0 \sum \gamma_\mu k_\mu \theta(k)\,\gamma_0\omega\psi(Pk), \psi'(Pk)\right\rangle d\Lambda(k).\end{aligned}$$

Making the change of variable $Pk \to k$, this becomes

$$\begin{aligned}&\int \left\langle \gamma_0\left(\gamma_0 k_0 - \sum \gamma_j k_j\right)\theta(k)\,\gamma_0\omega\psi(k), \gamma_0\omega\psi'(k)\right\rangle d\Lambda(k)\\ &\quad = \int \left\langle \omega^+\gamma_0{}^+\gamma_0\gamma_0(-\omega)\left(\sum \gamma_\mu k_\mu\right)\theta(k)\,\psi(k), \psi'(k)\right\rangle d\Lambda(k)\\ &\quad = \int \left\langle \omega\gamma_0^3(-\omega)\left(\sum \gamma_\mu k_\mu\right)\theta(k)\,\psi(k', \psi'(k)\right\rangle d\Lambda(k)\\ &\quad = \int \langle\!\langle \chi\psi(k), \psi'(k)\rangle\, d\Lambda(k).\end{aligned}$$

Thus $U(P)$ is antiunitary.

In the case of T,

$$U(T)\,\chi\colon \psi(k) \to -\gamma_0\omega(-\gamma_0 k_0 + \gamma_1 k_1 + \gamma_2 k_2 + \gamma_3 k_3)(-\theta(k))\,\psi(Tk)$$

since $\theta(Tk) = -\theta(k)$, or

$$\gamma_0\omega\left(-\gamma_0 k_0 + \sum \gamma_j k_j\right)\theta(k)\,\psi(Tk);$$

$$\chi U(T)\colon \psi(k) \to -\left(\sum \gamma_\mu k_\mu\right)\theta(k)\,\gamma_0\omega\psi(Tk)$$

$$= -\gamma_0\left(\gamma_0 k_0 - \sum \gamma_j k_j\right)\theta(k)\,\omega\psi(Tk) = \gamma_0\omega\left(\gamma_0 k_0 - \sum \gamma_j k_j\right)\theta(k)\,\psi(Tk).$$

Thus $U(T)$ and χ anticommute. The anti-unitarity of $U(T)$ now follows by essentially the same argument in the case of $U(P)$.

The action of C is trivial in the real formalism. There is in fact no operator that acts only in the spin space at each point, is relativistically covariant, and interchanges $\mathbf{S}_+$ and $\mathbf{S}_-$. To see this, let C denote such an operator. Then $C = C_0 + C_1 + C_2 + C_3 + C_4$, where C_m is a linear combination of products of m of the γ_μ. Relativistic covariance requires that C commute with all $\gamma_\mu\gamma_\nu$. The commutator $[C_m, \gamma_\mu\gamma_\nu]$ is again a linear combination of products of m of the γ_μ. Hence $[C_m, \gamma_\mu\gamma_\nu]$ must vanish for all m, μ, and ν. On the other hand, if m is even, C_m commutes with ω and η, thus leaves each of $\mathbf{S}_+$ and $\mathbf{S}_-$ invariant, and so must vanish. It suffices therefore to consider the two cases $m = 1$, $m = 3$. By Lorentz invariance, C_1 may be assumed to be of the form $a\gamma_\mu$, where a is a constant, which is not Lorentz invariant unless $a = 0$. The case $m = 3$ is similar.

However, the Dirac operator $D = \sum \gamma_\mu\,\partial_\mu$ anticommutes with ω and thus when $m > 0$ provides a relativistically covariant first-order differential operator that interchanges the $\mathbf{S}_\pm$. It is in fact the unique such operator apart from ωD. Note that D is invertible when $m > 0$: $D^2 = -m^2$ (on $\mathbf{S}$). It follows also that if $m > 0$, then the Dirac equation is not satisfied by non-vanishing wave functions in either $\mathbf{S}_+$ or $\mathbf{S}_-$. On the other hand, if $m = 0$, the Dirac equation is satisfied by the wave functions in $\mathbf{S}$. For the range of D applied to $\mathbf{S}_+$ is a $\mathbf{P}$-invariant subspace. By irreducibility, it is either just 0, or consists of one of the $\mathbf{S}_\pm$. Because of the anticommutativity of ω and D, it can not be $\mathbf{S}_+$, and because $D^2 = 0$, it can not be $\mathbf{S}_-$, since by symmetry it would also be the case that $D\mathbf{S}_+ = 0$, and hence is 0. Thus the Dirac equation follows in the massless case from the $\mathbf{P}$-transformation properties of the real spinors.

Temporal evolution in **S** is represented by a continuous one-parameter unitary group, with a selfadjoint generator H. The abstract first-order differential equation $i\,\partial_0\,\psi = H\psi$ is valid, and may be expressed as the local second-order differential equation $(\Box + m^2)\,\psi = 0$, for which the Cauchy (initial value) problem is well-posed. Thus the Klein–Gordon equation is satisfied by each component, and this suffices, in conjunction with real Cauchy data for ψ and $\partial_0\,\psi$ to determine ψ globally.

The fact that the same general formalism applies equally to massive and massless fields contrasts with conventional theory, but underscores the connection between e and ν and strengthens so-called "weak isospin."

It is well-known that the neutrino field may be directly represented in terms of Majorana fields. The difference from conventional formalism is that the electron field is here entirely parallel. This contrasts strongly with the conception dating from the discovery of parity nonconservation that there was a fundamental and theoretically unexpected qualitative difference between the two types of fields, quite apart from the mass difference.

4. EIGENSTATES OF THE BASIC QUANTUM NUMBERS

To exemplify the real formalism, the electron eigenstates for energy-momentum and helicity will be given explicitly. It suffices to do so for the case of fields in $\mathbf{S}_+$, since the case of $\mathbf{S}_-$ is similar.

The real spinor field of mass m, $\psi(x) = A\,\sin(k\cdot x) + B\cos(k\cdot x)$, where A and B are constant spinors and $k^2 = m^2$ will be in $\mathbf{S}_+$ if and only if $(\omega + \eta)\,\psi = 0$. The Hilbert transform η does not affect the spin components. When $k_0 > 0$, the action of η on complex exponentials is to send $\exp(ik\cdot x)$ into $i\exp(ik\cdot x)$, or in real terms:

$$\eta\colon \sin(k\cdot x) \to -\cos(k\cdot x),\ \cos(k\cdot x) \to \sin(k\cdot x).$$

Thus $\eta\psi = -A\cos(k\cdot x) + B\sin(k\cdot x)$, while $\omega\psi = \omega\,A\,\sin(k\cdot x) + \omega B\cos(k\cdot x)$, and the vanishing of $(\omega + \eta)\,\psi$ is equivalent to the equations $-A = \omega B$, $B = \omega A$, of which the first equation is redundant.

Let ψ now have the form $A\sin(k\cdot x) - \omega A\cos(k\cdot x)$. Then ψ is a state of energy-momentum k:

$$\omega\partial_\mu\psi = \omega[k_\mu A\,\sin(k\cdot x) - \omega A(-k_\mu\,\sin(k\cdot x)] = k_\mu\,\psi;$$

$$\begin{aligned}\psi(x-a) &= A\,\sin[k\cdot(x-a)] - \omega A\,\cos[k\cdot(x-a)]\\ &= A[\sin(k\cdot x)\cos(k\cdot a) - \cos(k\cdot x)\sin(k\cdot a)]\\ &\quad - \omega A[\cos(k\cdot x)\cos(k\cdot a) + \sin(k\cdot x)\sin(k\cdot a)]\end{aligned}$$

$$
\begin{aligned}
&= [A \sin(k \cdot x) - \omega A \cos(k \cdot x)] \cos(k \cdot a) \\
&\quad - \omega [A \sin(k \cdot x) - \omega A \cos(k \cdot x)] \sin(k \cdot a) \\
&= [\cos(k \cdot a) - \omega \sin(k \cdot a)][A \sin(k \cdot x) - \omega A \cos(k \cdot x)] \\
&= \exp[-\omega(k \cdot a)] \psi,
\end{aligned}
$$

noting that for any real number B, $\exp(B\omega) = \cos B + \omega \sin B$.

The general helicity eigenstate can be obtained by a Lorentz transformation, from that for the case in which the momentum is in the z-direction. In this case, the helicity h takes the form $-\omega^{-1}\gamma_1\gamma_2\theta(k_3) = \gamma_0\gamma_3\theta(k_3)$. It is straightforward to show that h has two real eigenvectors of eigenvalues ± 1, and that the corresponding eigenspinors are (resp.) of the form $A^+ = (a, b, b, a)$ and $A^- = (a, b, -b, -a)$. Thus $\mathbf{S}_+$ contains states labeled just as in the Dirac theory in terms of the basic quantum numbers.

When $m = 0$, the states are likewise identical to those of standard theory. For a massless wave function ψ in $\mathbf{S}_+$, the equation $Dy = 0$ implies that $\partial_0 \psi = (\sum \gamma_0 \gamma_j \partial_j) \psi$, from which it follows that the helicity in an energy-momentum eigenstate is ± 1. Analytically these states appear just as in the massive case given above.

5. INTERACTIONS IN THE REAL SPINOR FORMALISM

As seen above, massive and massless spinor particles are on very similar footings in real fields, and it is natural to postulate that the interactions of electrons and neutrinos are parallel. The simplest form is a trilinear interaction involving a intermediary boson together with a fermion—antifermion current.

The electromagnetic interaction lagrangian then takes the form $L = \langle\!\langle A\psi, \psi \rangle\!\rangle$, where ψ is the *real* spinor field of mass $m > 0$, and $A = \sum_\mu A_\mu \gamma_\mu$, where the one-form photon field is $\sum_\mu A_\mu \, dx_\mu$. In terms of complex spinors, this is equivalent to discarding the right current from the conventional $j_\mu A_\mu$ interaction, and retaining only the left current. The conventional current is simply the sum of these two currents. In the corresponding absence of any mixing of these currents, no observable difference from the implications of complex QED regarding states representable by real massive spinor fields can appear. All electron states distinguishable by the conventional quantum numbers of energy, momentum and helicity are included among the real fields.

In particular, modulo the practical necessity of heuristic renormalizations in Minkowski space, the observable predictions of the real formalism regarding electromagnetic phenomena, from Compton scattering to the

Lamb shift, would be analytically identical to those of the conventional complex formalism. In the absence of any known direct experimental indications in electromagnetic phenomena for the physical reality of additional modes, beyond those corresponding uniquely to the quantum numbers given above, this is consistent with the physical redundancy of half of the complex massive spinor modes. Because the $W^{\pm}$ couplings in the conventional model are purely $V-A$, they are entirely equivalent to those obtained in the real formalism, the right electron being effectively automatically suppressed.

At first glance, it might appear that $U(1)$ gage invariance is lost in the real formalism for QED. However, with the redefinition of gauge transformations on spinor fields as $\psi \to \exp(\omega\alpha(x)\,\psi$, and a corresponding change in the fermion lagrangian, it proceeds essentially unchanged. The modified lagrangian takes the form $\langle\!\langle \omega(\sum \gamma_\mu \partial_\mu)\,\psi, \psi \rangle\!\rangle - m\langle\!\langle \psi, \psi \rangle\!\rangle$, and so is local and **P**-covariant. The boson lagrangians require no change, while the interaction lagrangian is formally unaffected apart from the substitution of ω where conventionally i appears, but as indicated above, is restricted to real spinors.

6. DISCUSSION

Conventional neutral current theory is only very roughly approximated, but because of its dependence on divergent renormalization prescriptions for large corrections in order to reach agreement with experiment it can not be considered quite satisfactory. As noted by Marciano [8], "[correlation with] experiment requires a renormalization prescription and complete $O(\alpha)$ calculation of radiative corrections." The basis for assessing the adequacy of $O(\alpha)$ corrections at the very high energies involved remains fundamentally uncertain, in view of the divergences and the total absence of bounds on higher-order effects. The large radiative and other model dependent corrections used in a variety of precision tests of the conventional model are beyond the scope of independent substantiation, and their correctness can not be inferred simply from the seeming empirical validity of the corrected results. In addition, strong interaction effects may, as in other situations, disturb some of the neutral current experiments.

The real formalism is free of divergences when transferred from Minkowski space to the universal cosmos, as shown by essentially the same argument as that in the case of QED [9], in conjunction with the harmonic analysis by Paneitz of all positive-energy vector fields [10]. It promises to provide an economical and comprehensive particle theory embracing particles of weight 5/2 dual to those of weight 3/2, which may be involved in neutral current interactions, and for whose existence there is evidence in astronomical observation as well as particle experiments. Correlation of the elementary particle theory based on real fields in the universal cosmos [1] with observed particles is outlined in [11], and with cosmic considerations in [12]. In any event, the real formalism may provide a significant clue to a theory in Minkowski space beyond the so-called "standard model."

In summary, the real formalism provides an analytically economical alternative to conventional theory that clarifies the origin of $V-A$, enhances weak isospin, and leaves essentially intact the electromagnetic and charged current part of the practical physical theory.

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